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Discounted perpetual game put options

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ABSTRACT

The aim of this study is to explore the behavior of perpetual game put options, also known as cancellable puts. Their main characteristic is the opportunity of the buyer and the seller to exercise prematurely. If the seller decides to terminate the option, he obliges to pay a penalty amount above the normal option fee. We include also a discount factor that provides an advantage for earlier option exercising. We obtain the optimal moments for both participants to end the option promptly. This allows us to turn the option pricing problem to a first exit problem. We base our examination on financial instruments with random maturities. These instruments permit one of the partakers to maximize his expected future profit.

1. Introduction

The American style derivatives are an important class of financial instruments. Differently from their European analogues, the American derivatives can be exercised at every moment before the maturity. Thus their holder can insure the underlying asset for the whole period. The growing scientific interest to such derivatives is supported by the large number of publications in the recent years – we refer to Park and Jeon [1], Kang et al. [2], Zhao and Yang [3], Soleymani et al. [4], Chan [5], Detemple et al. [6], Armerin [7], Li and Ye [8], Magirou et al. [9], Arregui et al. [10], Yang [11], Zaevski [12], Gao et al. [13], Lu and Putri [14]. Differently from the option's buyer, the seller has no rights – he has only the obligation to fulfill the contract when the buyer decides to exercise. On the contrary, the game options give an early canceling right to the option's seller too. He can terminate the contract during the option life paying some penalty compensation which we assume to be a constant.

Together with Kifer [15], several authors examine the behavior of the cancellable puts. Ekström [16] considers the optimal regions for a game put option using its value function. Pricing of such options is viewed as a free boundary problem in Baurdoux and Kyprianou [17]. Kallsen and Kühn [18] and Kühn [19] examine an incomplete market. Some other examples of game put options are presented in Kyprianou [20], and Suzuki and Sawaki [21]. A path-wise approach can be found in Kühn et al. [22]. Hedging strategies for evaluating the game options are used in Hamadène [23] and Y. Dolinsky and Kifer [24]. The relation between the game options and the backward stochastic differential equations can be find in Matoussi et al. [25]. Discounted perpetual game call options are examined in Zaevski [26], whereas options with a proportional penalty are studied in Zaevski [27].

In addition to the classical form of the option contracts we introduce a discount factor. Another important characteristic we impose is the lapse of maturity. In such a way the supplement of discounting is the only factor which gives advantages for earlier exercising. Also, it is well known that a model with a continuous dividend payment can be described as a non-dividend model with discounted payments.[1] Hence, we can assume that there are no dividends keeping the model generality.

It is accepted in the present literature, that the cancellable puts are simpler for examining than the corresponding calls. This is due to the fact that the seller's exercise region is either the empty set or consists only of the strike. This is right, but not entirely. More

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[1] This is proven in Zaevski [26].

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concretely, it is true when the risk free rate is positive. When it is negative, the seller's optimal region can be as complicated as the call style seller's region. It may seem that the negative risk free rate is unnatural, but it appears when we convert a model with dividend payments to a non-dividend one. More precisely, when the dividend rate is larger than the risk free rate.

The approach we use to price game options is based on the so called exercise boundaries. They are these critical values for the underlying asset which make keeping the option more preferable than the immediate contract implementation. If the asset price falls enough it can be optimal for the buyer to exercise the option. Otherwise, if the asset is near to the strike and the penalty is small enough it can be optimal for the seller to cancel the option. It turns out that these boundaries are some constants due to the Markovian property of the underlying asset combined with the lapse of a date at which the contract expires. We prove that the area where the immediate exercise is optimal for the buyer is a strip, whereas the seller's optimal region can be a strip, a singleton or even the empty set.

To identify the exercise boundaries we use American style derivatives with a random expiration date. Once we know the boundaries and the corresponding regions we can use the first exit and hitting properties to determine the option price.

Most of the results we derive are true for the finite maturity case too. We confined our research assuming that the asset price process is log-normal, but some of the results are true for an arbitrary Markov process too.

We organize the paper in the following way. In Section 2 we state our model and prove some propositions related to the optimal regions. The pricing results for the game put options are presented in Section 3. We validate our scheme by some numerical examples in Section 4. The necessary propositions related to the first exit time of the Brownian motion are presented in Appendix A. We prove in Appendix B that the equations for the exercise boundaries lead to unique values.

2. Exercise regions

Let the asset price be presented as a Markov process under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$. We shall work directly under the risk-neutral measure Q. We assume that the risk-free and discount rates are the constants r and λ, respectively. We mentioned above that after a change of the parameters values a model with dividends can be stated as a non-dividend model. This fact is proven as Proposition 2.3 in Zaevski [26]. Some values of the initial parameters can lead to a negative value for the risk-free rate. Hence, the case $r < 0$ is admissible. There are two necessary conditions for the discount rate: (A) it has to be non-negative, $\lambda \geq 0$, and (B) the aggregate discount rate has to be positive $r + \lambda > 0$. Let the non-negative constant $\eta \geq 0$ presents the seller's cancellation penalty. We shall denote by the function $N_1(t, x)$ the amount which the option's seller has to pay if the buyer decides to exercise at moment t provided that $S_t = x$. Analogously, we denote by $N_2(t, x)$ the function which defines the amount which the seller owes if he decides to cancel the contract. In such a way the payment functions are

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(K-x)^+ \\ N_2(t,x) &= e^{-\lambda t}\Big((K-x)^+ + \eta\Big). \end{aligned} \tag{2.1}$$

Also, we impose the following natural assumption.

Condition 2.1. If the underlying asset starts from a higher value, then its future values are higher too.

Another important proposition which gives the option time dependence is proven in Zaevski [26] as Proposition 2.2. It allows us to examine only the case $t = 0$.

Proposition 2.1. *If the option price can be presented as $Y(t, S_t)$, then the function $Y(t, x)$ satisfies the condition $Y(t,x) = e^{-\lambda t} Y(0,x)$.*

Now we shall define the seller's and buyer's optimal strategies

Definition 2.1. Let $t < T$ be some future moments. We can think that the larger one presents a maturity. Let us denote by $\mathcal{T}_{[t,T]}$ the set of all stopping times with values between t and T. Let $\zeta \in \mathcal{T}_{[t,T]}$ be arbitrary. Then the seller's and buyer's optimal strategies are defined as follows.

1. The ζ-seller's optimal strategy is denoted by $A(\zeta; x)$ and it minimizes

$$E^{t,x}\begin{bmatrix} e^{-r(\zeta-t)}N_1\big(\zeta, S_\zeta\big)I_{\zeta \leq A(\zeta;\cdot)} \\ +e^{-r(A(\zeta;\cdot)-t)}N_2\big(A(\zeta;\cdot), S_{A(\zeta;\cdot)}\big)I_{A(\zeta;\cdot)<\zeta} \end{bmatrix}. \tag{2.2}$$

2. Analogously, the ζ-buyer's optimal strategy is defined as the stopping time $B(\zeta; x)$ which maximizes

$$E^{t,x}\begin{bmatrix} e^{-r(B(\zeta;\cdot)-t)}N_1\big(B(\zeta;\cdot), S_{B(\zeta;\cdot)}\big)I_{B(\zeta;\cdot)\leq \zeta} \\ +e^{-r(\zeta-t)}N_2\big(\zeta, S_\zeta\big)I_{\zeta<B(\zeta;\cdot)} \end{bmatrix}. \tag{2.3}$$

We shall denote by Υ^s and Υ^b the seller's and buyer's optimal regions, respectively. We define them as follows.

Definition 2.2.

1. The point (t, x) is optimal for the buyer, $(t, x) \in \Upsilon^b$, if for an arbitrary stopping time $\zeta_1 \in \mathcal{T}_{[t,T]}$,

$$N_1(t,x) \geq E^{t,x}\begin{bmatrix} e^{-r(\zeta_1-t)}N_1\big(\zeta_1, S_{\zeta_1}\big)I_{\zeta_1 \leq A(\zeta_1;\cdot)} \\ +e^{-r(A(\zeta_1;\cdot)-t)}N_2\big(A(\zeta_1;\cdot), S_{A(\zeta_1;\cdot)}\big)I_{A(\zeta_1;\cdot)<\zeta_1} \end{bmatrix}. \tag{2.4}$$

2. The point (t, x) is the seller's optimal, $(t, x) \in \Upsilon^s$, if for every stopping time $\zeta_2 \in \mathcal{T}_{[t,T]}$,

$$N_2(t,x) \leq E^{t,x}\begin{bmatrix} e^{-r(B(\zeta_2;\cdot)-t)}N_1\big(B(\zeta_2;\cdot), S_{B(\zeta_2;\cdot)}\big)I_{B(\zeta_2;\cdot)\leq \zeta_2} \\ +e^{-r(\zeta_2-t)}N_2\big(\zeta_2, S_{\zeta_2}\big)I_{\zeta_2<B(\zeta_2;\cdot)} \end{bmatrix}. \tag{2.5}$$

We shall prove now some propositions which describe the exercise regions Υ^b and Υ^s.

Proposition 2.2. *If $x > K$, then $x \notin \Upsilon^s$.*

Proof. Suppose that $x > K$ and let τ be the first moment at which the asset hits the strike. This strategy leads to a seller's payment $e^{-\lambda\tau}\eta$. Since $r + \lambda > 0$, we have for its present value $E\Big[e^{-r\tau}e^{-\lambda\tau}\eta\Big] < \eta$. Thus we can see that the strategy τ is more favorable for the seller than immediate canceling. □

Proposition 2.3. *If $x \in \Upsilon^b$ and $0 < y < x$, then $y \in \Upsilon^b$ too.*

Proof. Note that x has to be lower than the strike, $x < K$. Let us set the maturity at level T. Let $A(\zeta; x)$ be the ζ-seller's optimal strategy for a buyer strategy $\zeta \in \mathcal{T}[0, T]$. Using inequality (2.4) we derive

$$E^x\begin{bmatrix} e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+ I_{\zeta \leq A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)}\Big(\big(K-S_{A(\zeta;x)}\big)^+ + \eta\Big)I_{A(\zeta;x)<\zeta} \end{bmatrix} \leq (K-x).$$

Using that expectation (2.2) has a minimum for $A(\zeta;y)$ and condition 2.1, we obtain

$$
\begin{aligned}
&E^y\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+ I_{\zeta\le A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)}\left(\left(K-S_{A(\zeta;y)}\right)^+ +\eta\right) I_{A(\zeta;y)<\zeta}\end{array}\right]-(K-y)\\
&=E^y\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+ I_{\zeta\le A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)}\left(\left(K-S_{A(\zeta;y)}\right)^+ +\eta\right) I_{A(\zeta;y)<\zeta}\end{array}\right]\\
&+E^y\left[e^{-(r+\lambda)\zeta}e^{\lambda\zeta}S_\zeta I_{\zeta\le A(\zeta;y)}+e^{-(r+\lambda)A(\zeta;y)}e^{\lambda A(\zeta;y)}S_{A(\zeta;y)}I_{A(\zeta;y)<\zeta}\right]-K\\
&=E^y\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\max\left(\left(e^{\lambda\zeta}-1\right)S_\zeta+K, e^{\lambda\zeta}S_\zeta\right) I_{\zeta\le A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)}\max\left(\begin{array}{l}\left(e^{\lambda A(\zeta;y)}-1\right)S_{A(\zeta;y)}+K+\eta,\\ e^{\lambda A(\zeta;y)}S_{A(\zeta;y)}+\eta\end{array}\right) I_{A(\zeta;y)<\zeta}\end{array}\right]-K\\
&\le E^y\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\max\left(\left(e^{\lambda\zeta}-1\right)S_\zeta+K, e^{\lambda\zeta}S_\zeta\right) I_{\zeta\le A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)}\max\left(\begin{array}{l}\left(e^{\lambda A(\zeta;x)}-1\right)S_{A(\zeta;x)}+K+\eta,\\ e^{\lambda A(\zeta;x)}S_{A(\zeta;x)}+\eta\end{array}\right) I_{A(\zeta;x)<\zeta}\end{array}\right]-K\\
&< E^x\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\max\left(\left(e^{\lambda\zeta}-1\right)S_\zeta+K, e^{\lambda\zeta}S_\zeta\right) I_{\zeta\le A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)}\max\left(\begin{array}{l}\left(e^{\lambda A(\zeta;x)}-1\right)S_{A(\zeta;x)}+K+\eta,\\ e^{\lambda A(\zeta;x)}S_{A(\zeta;x)}+\eta\end{array}\right) I_{A(\zeta;x)<\zeta}\end{array}\right]-K\\
&=E^x\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+ I_{\zeta\le A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)}\left(\left(K-S_{A(\zeta;x)}\right)^+ +\eta\right) I_{A(\zeta;x)<\zeta}\end{array}\right]-(K-x)\le 0.
\end{aligned}
$$

The last step is to take the limit $T\to\infty$. □

Proposition 2.4. *If $x\in\Upsilon^s$ and $x<y\le K$, then $y\in\Upsilon^s$.*

Proof. Let again the maturity T be finite and $\zeta\in\mathcal{T}[0,T]$ be a seller's strategy. Let us denote by $B(\zeta;\,x)$ the corresponding ζ-optimal strategy for the buyer. Therefore inequality (2.5) gives us

$$
E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(K-S_{B(\zeta;x)}\right)^+ I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\left(\left(K-S_\zeta\right)^+ +\eta\right) I_{\zeta<B(\zeta;x)}\end{array}\right]\ge (K-x+\eta).
$$

Using similar arguments as those in the proof of proposition 2.3, we conclude

$$
\begin{aligned}
&E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(K-S_{B(\zeta;y)}\right)^+ I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\left(\left(K-S_\zeta\right)^+ +\eta\right) I_{\zeta<B(\zeta;y)}\end{array}\right]-(K-y+\eta)\\
&=E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(K-S_{B(\zeta;y)}\right)^+ I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\left(\left(K-S_\zeta\right)^+ +\eta\right) I_{\zeta<B(\zeta;y)}\end{array}\right]\\
&+E^y\left[e^{-(r+\lambda)B(\zeta;y)}e^{\lambda B(\zeta;y)}S_{B(\zeta;y)}I_{B(\zeta;y)\le\zeta}+e^{-(r+\lambda)\zeta}e^{\lambda\zeta}S_\zeta I_{\zeta<B(\zeta;y)}\right]-(K+\eta)\\
&=E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\max\left(\begin{array}{l}\left(e^{\lambda B(\zeta;y)}-1\right)S_{B(\zeta;y)}+K,\\ e^{\lambda B(\zeta;y)}S_{B(\zeta;y)}\end{array}\right) I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\max\left(\begin{array}{c}\left(e^{\lambda\zeta}-1\right)S_\zeta+K+\eta,\\ e^{\lambda\zeta}S_\zeta+\eta\end{array}\right) I_{\zeta<B(\zeta;x)}\end{array}\right]-(K+\eta)\\
&\ge E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\max\left(\begin{array}{l}\left(e^{\lambda B(\zeta;x)}-1\right)S_{B(\zeta;x)}+K,\\ e^{\lambda B(\zeta;x)}S_{B(\zeta;x)}\end{array}\right) I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\max\left(\begin{array}{c}\left(e^{\lambda\zeta}-1\right)S_\zeta+K+\eta,\\ e^{\lambda\zeta}S_\zeta+\eta\end{array}\right) I_{\zeta<B(\zeta;x)}\end{array}\right]-(K+\eta)\\
&> E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\max\left(\begin{array}{l}\left(e^{\lambda B(\zeta;x)}-1\right)S_{B(\zeta;x)}+K,\\ e^{\lambda B(\zeta;x)}S_{B(\zeta;x)}\end{array}\right) I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\max\left(\begin{array}{c}\left(e^{\lambda\zeta}-1\right)S_\zeta+K+\eta,\\ e^{\lambda\zeta}S_\zeta+\eta\end{array}\right) I_{\zeta<B(\zeta;x)}\end{array}\right]-(K+\eta)\\
&=E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(K-S_{B(\zeta;x)}\right)^+ I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\left(\left(K-S_\zeta\right)^+ +\eta\right) I_{\zeta<B(\zeta;x)}\end{array}\right]-(K-x+\eta)\ge 0.
\end{aligned}
$$

It remains only to take the limit $T\to\infty$ to finish the proof. □

Proposition 2.5. *When the risk free rate is positive, $r>0$, the seller's optimal region is either empty, $\Upsilon^s=\emptyset$, or it is the singleton $\Upsilon^s=\{K\}$.*

Proof. In Proposition 2.2 is proven that $(K,\infty)\cap\Upsilon^s=\emptyset$. Suppose now that $x<K$, $x\in\Upsilon^s$, and x is not the exercise boundary. Using Proposition 2.3 we conclude that there exists a constant $K_1<x$ which is not optimal for the buyer, $K_1\notin\Upsilon^b$. Let us denote by ζ the moment in which the underlying asset leaves the strip (K_1, K). Proposition 2.3 leads to $B(\zeta;\,x)>\zeta$. Hence

$$
\begin{aligned}
&E^x\left[e^{-(r+\lambda)\zeta}\left(K-S_\zeta+\eta\right)I_{\zeta\le B(\zeta;x)}+e^{-(r+\lambda)B(\zeta;x)}\left(K-S_\zeta\right)I_{B(\zeta;x)<\zeta}\right]\\
&\le E^x\left[e^{-r\zeta}\left(K-S_\zeta+\eta\right)\right]<K+\eta-x,
\end{aligned}
$$

which means that $x\notin\Upsilon^s$. □

Note that in the terminal case $\eta=0$ Propositions 2.3 and 2.4 still hold. The following proposition gives the form of the exercise regions in this case.

Proposition 2.6. *If $\eta = 0$ and $x < K$, then immediate exercising is optimal for one of the participants.*

Proof. Let the initial asset value belongs to the interval (A, B) for some constants, $A < x < B \le K$. Let us define a financial instrument which pays amount of $K - B$ if the asset leaves the strip (A, B) from the upper boundary. If the exit happens from the lower boundary, then the derivative has to pay amount of $K - A$. We shall denote its price by $f(A, B, x)$. Let us notate by $B(A, x)$ the upper boundary which minimizes the derivative value in the interval $[x, K]$ assuming that the lower boundary is A. Analogously, let $A(B, x)$ be the boundary which maximizes the derivative value in the interval $(0, x]$ provided that the upper boundary is B.

Suppose that the value x is not optimal for the seller, $x \notin \Upsilon^s$. Proposition 2.3 leads to the existence of $\overline{B} > x$ which satisfies

$$f\left(A\left(\overline{B}, x\right), \overline{B}, x\right) < K - x. \tag{2.6}$$

Using that $A(B, x)$ maximizes the derivative value, we see that

$$f\left(A, \overline{B}, x\right) \le f\left(A\left(\overline{B}, x\right), \overline{B}, x\right) < x - K \tag{2.7}$$

for every $A \in (0, x]$. Since the function $B(A, x)$ minimizes the derivative price, we can conclude that

$$f(A, B(A, x), x) \le f\left(A, \overline{B}, x\right) < x - K. \tag{2.8}$$

This means that $x \in \Upsilon^b$. □

3. Pricing of cancellable puts

Assume now that the asset price is presented by the geometric Brownian motion

$$dS_t = rS_t + \sigma S_t dB_t. \tag{3.1}$$

Proposition 2.3 shows that the form of the buyer's exercise region is $\Upsilon^b = (0, A]$, whereas Proposition 2.4 shows that the seller's exercise region has one of the following three forms – $\Upsilon^s = [B, K]$, $\Upsilon^s = \{K\}$, or $\Upsilon^s = \emptyset$.

Suppose that the seller's boundary is less than the strike, $B < K$ and the initial asset price is between the boundaries, $x \in [A, B]$. Thus the problem of fair pricing of cancellable puts turns to a first exit problem of a Brownian motion from a strip. Let us notate by τ_A and τ_B the first moments when the asset price reaches the levels A and B, respectively. Hence, the first exit moment of the asset from the strip $[A, B]$ is $\tau = \tau^A \wedge \tau^B$. Note that τ^A and τ^B are the first hitting times of the Brownian motion with drift

$$\psi = \frac{r}{\sigma} - \frac{\sigma}{2} \tag{3.2}$$

to the levels

$$\bar{A} = \frac{\ln A - \ln x}{\sigma} < 0$$

$$\bar{B} = \frac{\ln B - \ln x}{\sigma}. \tag{3.3}$$

Thus the price of the cancellable put turns to

$$\begin{aligned} f(A, B, x) &= E\left[\begin{array}{l} e^{-(r+\lambda)\tau^{\bar{B}}}\left(K - S_{\tau^{\bar{B}}} + \eta\right) I_{\tau^{\bar{B}} \le \tau^{\bar{A}}} \\ +e^{-(r+\lambda)\tau^{\bar{A}}}\left(K - S_{\tau^{\bar{A}}}\right) I_{\tau^{\bar{A}} < \tau^{\bar{B}}} \end{array}\right] \\ &= (K - B + \eta) E^x\left[e^{-(r+\lambda)\tau^{\bar{B}}} I_{\tau^{\bar{B}} \le \tau^{\bar{A}}}\right] + (K - A) E^x\left[e^{-(r+\lambda)\tau^{\bar{A}}} I_{\tau^{\bar{A}} < \tau^{\bar{B}}}\right]. \end{aligned} \tag{3.4}$$

From now on we shall use the following notations

$$\mu := \frac{\psi}{\sigma},$$

$$c := \sqrt{\mu^2 + 2\frac{r+\lambda}{\sigma^2}} \equiv \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}}, \tag{3.5}$$

$q = c + \mu$, and $p = 2c$. Obviously, $c \ge \mu + 1$ or equivalently $p \ge q + 1$. We have equality in the undiscounted case $\lambda = 0$. Note that $r > 0$ if $\lambda = 0$. The Laplace transforms of the first exit time of the Brownian motion from a strip are given by Eqs. (A.3) and (A.4).[2] Thus we conclude that Eq. (3.4) transforms to

$$\begin{aligned} f(A, B, x) &= (K - A) e^{\psi \bar{A}} \frac{\sinh\left(\sigma c \bar{B}\right)}{\sinh\left(\sigma c\left(\bar{B} - \bar{A}\right)\right)} \\ &\quad + (K - B + \eta) e^{\psi \bar{B}} \frac{\sinh\left(-\sigma c \bar{A}\right)}{\sinh\left(\sigma c\left(\bar{B} - \bar{A}\right)\right)} \\ &= (K - A) e^{(c-\mu)(\ln x - \ln A)} \frac{e^{2c(\ln B - \ln x)} - 1}{e^{2c(\ln B - \ln A)} - 1} \\ &\quad + (K - B + \eta) e^{(c+\mu)(\ln B - \ln x)} \frac{e^{2c(\ln x - \ln A)} - 1}{e^{2c(\ln B - \ln A)} - 1} \\ &= (K - A)\left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^p - A^p} + (K - B + \eta)\left(\frac{B}{x}\right)^q \frac{x^p - A^p}{B^p - A^p}. \end{aligned} \tag{3.6}$$

Let us fix the upper boundary B. We shall use the following change of variables $a = \frac{A}{B}$, $k = \frac{K}{B}$, $y = \frac{x}{B}$, and $\xi = \frac{\eta}{B}$. Hence $0 < a < 1 < k$. Thus Eq. (3.6) transforms to

$$f(A, B, x) = \frac{B}{y^q} \frac{(k - a) a^q (1 - y^p) + (k - 1 + \xi)(y^p - a^p)}{1 - a^p}. \tag{3.7}$$

We have to derive the value of the variable a which maximizes the function

$$\begin{aligned} g(a; y) &= \tfrac{(k-a)a^q(1-y^p)+(k-1+\xi)(y^p-a^p)}{1-a^p} \\ &= \tfrac{-a^p(k-1+\xi)-a^{q+1}(1-y^p)+a^q k(1-y^p)+y^p(k-1+\xi)}{1-a^p}. \end{aligned} \tag{3.8}$$

Calculating its derivative, we obtain

$$g_a(a; y) = \frac{1 - y^p}{(1 - a^p)^2} a^{q-1}\left[\begin{array}{l} -a^{p+1}(p - q - 1) + a^p k(p - q) \\ -a^{p-q} p(k - 1 + \xi) - a(q + 1) + qk \end{array}\right]. \tag{3.9}$$

We show in Appendix B.2 that the equation

$$\begin{aligned} &-a^{p+1}(p - q - 1) + a^p k(p - q) \\ &-a^{p-q} p(k - 1 + \xi) - a(q + 1) + qk = 0 \end{aligned} \tag{3.10}$$

has a unique root in the interval $(0, 1)$ which we shall notate as $a(B)$. It is independent of the variable y and depends on B via k and ξ. The function $g(a; y)$ has a maximum in this point, since derivative (3.9) is positive before the root and negative after it. Hence, the buyer's strategy has to be the first hitting time to the value $A = a(B)B$ if the seller waits the asset to hit the level B.

The next step is to examine the exercise boundary of the seller. Let us fix the lower boundary A and make an analogous change of the variables – $b = \frac{B}{A}$, $k = \frac{K}{A}$, $y = \frac{x}{A}$, and $\xi = \frac{\eta}{A}$. We have that $1 < b$. Hence, Eq. (3.6) transforms to

$$f(A, B, x) = \frac{A}{y^q} \frac{(k - 1)(b^p - y^p) + (k - b + \xi) b^q (y^p - 1)}{b^p - 1}. \tag{3.11}$$

The b-derivative of the function $g(\cdot\,;\,\cdot)$,

$$g(b; y) = \frac{(k - 1)(b^p - y^p) + (k - b + \xi) b^q (y^p - 1)}{b^p - 1}, \tag{3.12}$$

is

$$\begin{aligned} g_b(b; y) &= \frac{y^p - 1}{(b^p - 1)^2} b^{q-1} \\ &\quad \times \left[\begin{array}{l} b^{p+1}(p - q - 1) - b^p(k + \xi)(p - q) \\ +b^{p-q} p(k - 1) + b(q + 1) - q(k + \xi) \end{array}\right]. \end{aligned} \tag{3.13}$$

Suppose that the discount factor really exists, i.e. $\lambda > 0$. We show in Appendix B.1 that the equation

[2] (3.0.5 a & b) from Borodin and Salminen [28].

$$b^{p+1}(p-q-1) - b^{p}(k+\xi)(p-q) + b^{p-q}p(k-1) + b(q+1) - q(k+\xi) = 0 \tag{3.14}$$

has a unique root larger than 1 and we shall notate it by $b(A)$. Function (3.11) has its minimum in it. Hence, if the buyer exercises when the underlying asset reaches the level A, then the seller's best strategy is to cancel the contract at the first time when the asset takes value $B = b(A)A$. Ergo, we can derive our candidate for the seller's exercise boundary solving the equation

$$a(y)b(ya(y)) = 1. \tag{3.15}$$

Accordingly, we obtain the buyer's optimal boundary as $A = a(B)B$, where B is the root of Eq. (3.15).

First, we shall investigate the case when the received value of the seller's boundary is below the strike, $B < K$. In such a way the optimal regions take the form $\Upsilon^b \equiv [0, A)$ and $\Upsilon^s \equiv [B, K]$. We conclude that: (1) if the initial asset price is less than the buyer's boundary, $x < A$, then the option price is $K - x$; (2) if $x \in [A, B]$, the option price is obtained by Eq. (3.6); (3) if $x \in [B, K]$, then the option price is $K - x + \eta$; and (4) if $K < x$, then the seller cancels when the asset reaches the strike and therefore the option price is

$$E^x\left[e^{-(r+\lambda)\tau}\left((S_\tau - K)^+ + \eta\right)I_{\tau<\infty}\right] = \eta E^x\left[e^{-(r+\lambda)\tau}I_{\tau<\infty}\right] = \eta\left(\frac{K}{x}\right)^q. \tag{3.16}$$

The last formula is obtained using Eq. (A.2) from Proposition A.1.

It may happen that the solution of Eq. (3.15) is larger than the strike, because we had changed the seller's payment from $(K-B)^+ + \eta$ to $K - B + \eta$ in formula (3.4). We had done this to make it differentiable. We prove in Proposition 2.2 that it is never optimal for the seller to cancel the option if the initial asset price is above the strike. Hence, the true seller's optimal region is either the singleton $\{K\}$ or the empty set. We have to recognize which case is actual. The seller has two choices – either to cancel the option when the underlying asset hits the strike or to do nothing. In the second case the cancellable put turns to an ordinary American put. In Zaevski [12] is shown that its optimal boundary is

$$A^* = \frac{q}{q+1}K. \tag{3.17}$$

Assuming that the asset starts from the strike, $x = K$, we see that the American put price is[3]

$$\overline{\eta} = \frac{K}{q+1}\left(\frac{q}{q+1}\right)^q. \tag{3.18}$$

If the seller chooses to cancel the option immediately he has to pay amount η. So, we can see that its optimal region is the singleton $\{K\}$ when $\eta < \overline{\eta}$, and it is the empty set otherwise. In the first case we can proceed similarly to the case $B < K$, whereas in the second case we have an ordinary American put.

Suppose now that the game option is without discounting, i.e. $\lambda = 0$. Thus we have $r > 0$. In appendix B.1 we prove that derivative (3.13) is negative in the whole interval $(1, \infty)$. This means that price function (3.11) is minimized for $B = \infty$. Hence, it is actual the case when the derived boundary is larger than the strike. This corresponds to Proposition 2.5. We had examined this case above.

We resume the obtained results for the cancellable puts in the following theorem.

Theorem 3.1. *Let the boundaries which produces our algorithm be $\overline{A}$ and $\overline{B}$. We shall denote by Y the price of the cancellable put. We have*

1. *When $\overline{B} < K$, the optimal regions are $\Upsilon^s = \left[\overline{B}, K\right]$ and $\Upsilon^b = \left(0, \overline{A}\right]$. The price of the cancellable put is obtain as*
   (a) *If $x \geq K$, then*
   $$Y = \eta\left(\frac{K}{x}\right)^q. \tag{3.19}$$
   (b) *If $\overline{B} < x < K$, then*
   $$Y = K - x + \eta. \tag{3.20}$$
   (c) *If $\overline{A} \leq x \leq \overline{B}$, then*
   $$Y = \left(K - \overline{A}\right)\left(\frac{\overline{A}}{x}\right)^q \frac{\overline{B}^p - x^p}{\overline{B}^p - \overline{A}^p} + \left(K - \overline{B} + \eta\right)\left(\frac{\overline{B}}{x}\right)^q \frac{x^p - \overline{A}^p}{\overline{B}^p - \overline{A}^p}. \tag{3.21}$$
   (d) *If $x < \overline{A}$, then*
   $$Y = K - x. \tag{3.22}$$
2. *If $K \leq \overline{B}$, then*
   (a) *if $\eta \geq \overline{\eta}$, then the cancellable put turns to an ordinary American put and the exercise regions turn to $\Upsilon^s = \emptyset$ and $\Upsilon^b = (0, A^*]$. Equation (3.17) determines the buyer's optimal boundary A^*. If the asset starts below this value, $x \in \Upsilon^b \equiv (0, A^*]$, then the option price is (3.22). Otherwise, we derive it as*
   $$Y = \left(\frac{q}{x}\right)^q\left(\frac{K}{q+1}\right)^{q+1}. \tag{3.23}$$
   (b) *When $\eta < \overline{\eta}$, the seller's optimal region is a singleton, $\Upsilon^s = \{K\}$. The corresponding buyer's exercise region is $\Upsilon^b = \left(0, \overline{A}\right]$, where the value of $\overline{A}$ is derived as $\overline{A} = Ka(K)$ where $a(K)$ is the root of function (3.13).*
   If the asset starts above the strike, then formula (3.19) determines the option price. If $x \in \left(\overline{A}, K\right)$, then formula (3.21) gives the option price. Finally, if the asset starts from the buyer's exercise region, $x \leq \overline{A}$, then the price of the option is obtained by formula (3.22).

Remark 3.1. We had proved in Proposition 2.5 that if the risk free rate is positive, then the seller's exercise region is either empty or it is the singleton $\{K\}$. Hence, the second case of Theorem 3.1 holds.

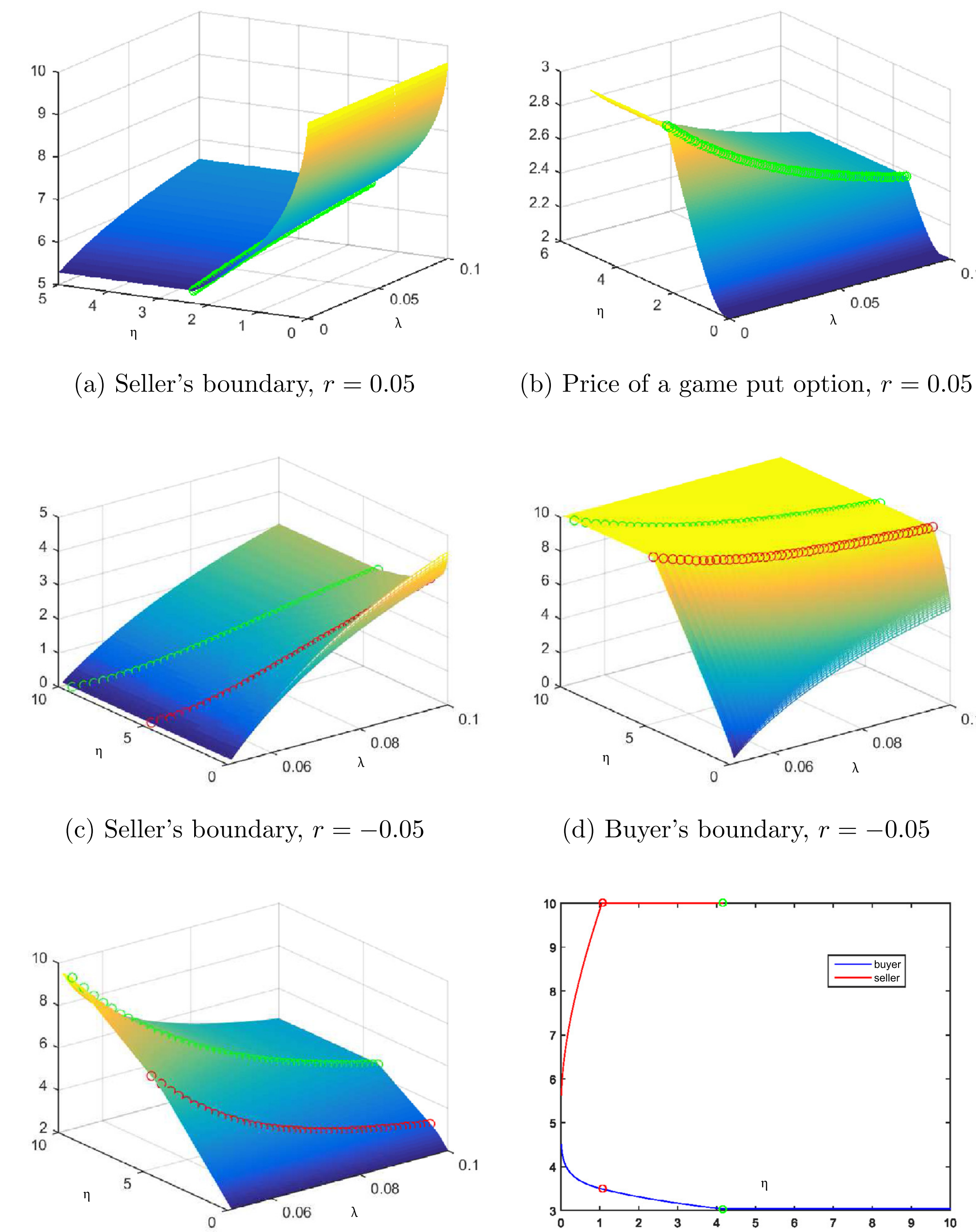
4. Numerical results

We discuss in this section the behavior of the cancellable puts. As we mentioned above, it is quite different if the risk free rate is positive or negative. First we suppose that it is positive and has a value $r = 0.05$. This means that the seller's optimal region consists only of the strike or is empty. The other parameters are $\sigma = 0.3$, $K = \$10$, and $x = \$8$. The penalty amount is set to be between zero and \$5. We vary the discount rate between 0.001 and 0.1. We present the results at Fig. 1. At the first graphic, sub-Fig. 1a, is presented the buyer's optimal boundary, whereas at sub-Fig. 1b is presented the behavior of the option prices. With green points are marked the critical values $\overline{\eta}$ given by Eq. (3.18). When the penalty is larger than $\overline{\eta}$, the cancellable puts turn to ordinary American puts.

For some particular values we present the option prices in Table 1. We place right to the prices the case of Theorem 3.1 which holds. We can see that the prices are decreasing w.r.t. the initial asset value, whereas they are non-decreasing w.r.t. the penalty η. Also, after the critical value the option prices do not depend on the penalty, because for a large enough penalty, the options become American.

We examine also the case when the risk free rate is negative, $r = -0.05$. Since $r + \lambda > 0$, we vary the discount rate λ between 0.051 and 0.1. Differently to the positive case, we examine penalties in the interval $\eta \in (0, 10)$. We mark by red the values of the

[3] See again Zaevski [12].



(a) Seller's boundary, $r = 0.05$

(b) Price of a game put option, $r = 0.05$

(c) Seller's boundary, $r = -0.05$

(d) Buyer's boundary, $r = -0.05$

(e) Price of a game put option, $r = -0.05$

(f) Both boundaries, $r = -0.05$, $\lambda = 0.1$

Fig. 1. The optimal boundaries and the option prices: $r = \pm 0.05$, $\sigma = 0.03$, $K = 10$, $S_0 = 8$, $\lambda \in (0.001, 0.1)$ ($\lambda \in (0.051, 0.1)$ if $r = -0.05$), $\eta \in (0, 5)$($\eta \in (0, 10)$ if $r = -0.05$). At Fig. 1e the discount factor is $\lambda = 0.1$. The red and green points are the penalty critical values at which the option changes its behavior. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

penalty η above which the boundary B is larger than the strike K. In such a way the actual case of Theorem 3.1 changes from the first to the second one. The points marked by green presents again the values of $\overline{\eta}$ from Eq. (3.18). In that way before the red points the seller's optimal region is the strip $[B, K]$, between the red and green points it is the singleton $\{K\}$, and after the green points it is the empty set and thus the game option turns to an ordinary American put. At sub-Fig. 1e are presented jointly the seller's and buyer's boundaries when $\lambda = 0.1$. The corresponding critical values (marked again by red and green) are 1.0537 and 4.1404, respectively. We can also find a validation of Proposition 2.6 in this sub-figure – when the penalty η tends to zero, both boundaries tend to one and the same value. We give the price behavior of the cancellable puts in sub-Fig. 1d. In Table 2 are reported the option prices for the following values of the parameters. The initial asset price is taken to be $x \in \{7, 8, 9, 10, 11\}$, the penalty is among $\eta \in \{0.001, 1, 2, 3, 4\}$, and the discount rates are $\lambda \in \{0.051, 0.06, 0.15\}$. Note that $r + \lambda > 0$. We can see that for large values of the penalty, the derived seller's boundary is above the strike and therefore the second case of Theorem 3.1 holds. Also, the cancellable puts turn to ordinary American puts when the penalty is sufficiently large.

Table 1
Put option prices: $r = 0.05$. To the right of the price is the actual case of Theorem 3.1.

	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
Initial price $S_0 = 7$					
$\lambda = 0.001$	3.0000/2.a	3.0245/2.b	3.3163/2.b	3.4402/1.b	3.4402/1.b
$\lambda = 0.01$	3.0000/2.a	3.0193/2.b	3.2915/2.b	3.3599/1.b	3.3599/1.b
$\lambda = 0.1$	3.0000/2.a	3.0000/2.a	3.0445/1.b	3.0445/1.b	3.0445/1.b
Initial price $S_0 = 8$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	2.0000/2.a	2.2224/2.b	2.7722/2.b	2.9617/1.b	2.9617/1.b
$\lambda = 0.01$	2.0000/2.a	2.2131/2.b	2.7493/2.b	2.8580/1.b	2.8580/1.b
$\lambda = 0.1$	2.0000/2.a	2.1431/2.b	2.3679/1.b	2.3679/1.b	2.3679/1.b
Initial price $S_0 = 9$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	1.0000/2.a	1.5622/2.b	2.3456/2.b	2.5952/1.b	2.5952/1.b
$\lambda = 0.01$	1.0000/2.a	1.5560/2.b	2.3320/2.b	2.4779/1.b	2.4779/1.b
$\lambda = 0.1$	1.0000/2.a	1.5045/2.b	1.8971/1.b	1.8971/1.b	1.8971/1.b
Initial price $S_0 = 10$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	0.0010/2.c	1.0000/2.c	2.0000/2.c	2.3060/1.b	2.3060/1.b
$\lambda = 0.01$	0.0010/2.c	1.0000/2.c	2.0000/2.c	2.1810/1.b	2.1810/1.b
$\lambda = 0.1$	0.0010/2.c	1.0000/2.c	1.5559/1.b	1.5559/1.b	1.5559/1.b
Initial price $S_0 = 11$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	0.0009/2.d	0.8986/2.d	1.7972/2.d	2.0722/1.b	2.0722/1.b
$\lambda = 0.01$	0.0009/2.d	0.8909/2.d	1.7819/2.d	1.9431/1.b	1.9431/1.b
$\lambda = 0.1$	0.0008/2.d	0.8358/2.d	1.3004/1.b	1.3004/1.b	1.3004/1.b

Table 2
Put option prices: $r = -0.05$. To the right of the price is the actual case of Theorem 3.1.

	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
Initial price $S_0 = 7$					
$\lambda = 0.051$	3.0010/1.b	4.0000/1.b	5.0000/1.b	6.0000/1.b	6.8529/1.c
$\lambda = 0.06$	3.0010/1.b	4.0000/1.b	4.9501/1.c	5.5321/2.b.2	5.9839/2.b.2
$\lambda = 0.15$	3.0010/1.b	3.3274/2.b.2	3.6184/2.b.2	3.9151/2.a.2	3.9151/2.a.2
Initial price $S_0 = 8$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.051$	2.0010/1.b	3.0000/1.b	4.0000/1.b	5.0000/1.b	5.9827/1.c
$\lambda = 0.06$	2.0010/1.b	3.0000/1.b	4.0000/1.b	4.7835/2.b.2	5.3922/2.b.2
$\lambda = 0.15$	2.0010/1.b	2.5763/2.b.2	3.0559/2.b.2	3.5321 /2.a.2	3.5321 /2.a.2
Initial price $S_0 = 9$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.051$	1.0010/1.b	2.0000/1.b	3.0000/1.b	4.0000/1.b	5.000/1.b
$\lambda = 0.06$	1.0010/1.b	2.0000/1.b	3.0000/1.b	3.9413/2.b.2	4.7325/2.b.2
$\lambda = 0.15$	1.0010/1.b	1.8114/2.b.2	2.5253/2.b.2	3.2255/2.a.2	3.2255/2.a.2
Initial price $S_0 = 10$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.051$	0.0010/1.b	1.0000/1.b	2.0000/1.b	3.0000/1.b	4.0000/1.b
$\lambda = 0.06$	0.0010/1.b	1.0000/1.b	2.0000/1.b	3.0000/2.b.2	4.0000/2.b.2
$\lambda = 0.15$	0.0010/1.b	1.0000/2.b.2	2.0000/2.b.2	2.9738/2.a.2	2.9738/2.a.2
Initial price $S_0 = 11$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.051$	0.0010/1.a	0.9990/1.a	1.9980/1.a	2.9970/1.a	3.9960/1.a
$\lambda = 0.06$	0.0010/1.a	0.9905/1.a	1.9809/1.a	2.9714/2.b.1	3.9619/2.b.1
$\lambda = 0.15$	0.0009/1.a	0.9291/2.b.1	1.8583/2.b.1	2.7631/2.a.2	2.7631/2.a.2

5. Conclusion

In this article we examined cancellable puts. We introduced also a discount factor which allows us to price options whose underlying asset pays a continuous dividend. We first derived the early exercise regions for both participants. It turns out that these regions strongly depend on the particular values of the parameters. We derived the equations, which the exercise boundaries satisfy, maximizing the buyer's and seller's financial expectations. We proved that these equations have unique roots. The obtained results for the exercise regions and the corresponding option prices are presented in Theorem 3.1. We gave as examples some numerical results. The parameters values are chosen in a way to include all different cases – positive and negative risk-free rates, large and small penalties and discount rates. All of the reported results validate the consistency of the proposed pricing model.

Declaration of Competing Interest

All authors have participated in (a) conception and design, or analysis and interpretation of the data; (b) drafting the article or revising it critically for important intellectual content; and (c) approval of the final version.

- This manuscript has not been submitted to, nor is under review at, another journal or other publishing venue.
- The authors have no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript

CRediT authorship contribution statement

Tsvetelin S. Zaevski: Conceptualization, Methodology, Formal analysis, Software, Investigation, Writing - original draft, Writing - review & editing.

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Appendix A. First exit propositions

Proposition A.1. *The Laplace transform of the first hitting time, τ, of a Brownian motion with drift μ to the level a is*

$$E\left[e^{-y\tau}I_{\tau<\infty}\right] = e^{-\left(\sqrt{\mu^2+2y}-\mu\right)a} \tag{A.1}$$

when $a > 0$, and

$$E\left[e^{-y\tau}I_{\tau<\infty}\right] = e^{\left(\sqrt{\mu^2+2y}+\mu\right)a} \tag{A.2}$$

when $a < 0$.

Proposition A.2. *Let us notate by τ the first exit time of a Brownian motion with drift μ from a strip (a, b). Then its Laplace transforms are given by the formulas*

$$E\left[e^{-y\tau}I_{\tau=a}\right] = e^{\mu a}\frac{\sinh\left(b\sqrt{2y+\mu^2}\right)}{\sinh\left((b-a)\sqrt{2y+\mu^2}\right)} \tag{A.3}$$

$$E\left[e^{-y\tau}I_{\tau=b}\right] = e^{\mu b}\frac{\sinh\left(-a\sqrt{2y+\mu^2}\right)}{\sinh\left((b-a)\sqrt{2y+\mu^2}\right)}. \tag{A.4}$$

Appendix B. Uniqueness of the solutions

B1. Seller's boundary

We shall prove that, except in the undiscounted case, derivative (3.13) has just one root larger than one. Let the function $h(\,\cdot\,;\,\cdot\,)$ be

$$h(b;\xi) = b^{p+1}(p-q-1) - b^p(k+\xi)(p-q) + b^{p-q}p(k-1) + b(q+1) - q(k+\xi). \tag{B.1}$$

We shall examine first the undiscounted case, $\lambda = 0$. We know that $p = q+1$. Thus function (B.1) transforms to

$$h(b;\xi) = -b^{q+1}(k+\xi) + b(q+1)k - q(k+\xi). \tag{B.2}$$

Its first derivative is

$$h_b(b;\xi) = -b^q(q+1)(k+\xi) + (q+1)k. \tag{B.3}$$

Obviously, it is negative in the whole interval $b \in (1, \infty)$ and therefore the function $h(b;\xi)$ is decreasing. Therefore the equation $h(b;\xi) = 0$ has no roots larger than one, since $h(1;\xi) = -p\xi < 0$.

Assume now that $p > q+1$. We change the variable as $d = \frac{1}{b}$. Thus the function $h(b;\xi)$ turns to

$$h(d;\xi) = \frac{1}{d^{p+1}}\begin{bmatrix} -d^{p+1}q(k+\xi) + d^p(q+1) + d^{q+1}p(k-1) \\ -d(k+\xi)(p-q) + (p-q-1) \end{bmatrix}. \tag{B.4}$$

Let us define the function $\bar{h}(\cdot;\cdot)$ as

$$\bar{h}(d;\xi) = -d^{p+1}q(k+\xi) + d^p(q+1) + d^{q+1}p(k-1) - d(k+\xi)(p-q) + (p-q-1). \tag{B.5}$$

Its first derivative is

$$\bar{h}_d(d;\xi) = -d^p(p+1)q(k+\xi) + d^{p-1}p(q+1) + d^q(q+1)p(k-1) - (k+\xi)(p-q). \tag{B.6}$$

We shall continue with the case $\xi = 0$. Note that $\bar{h}_d(d;\xi) < \bar{h}_d(d;0)$. We have for the second derivative of function (B.5)

$$\bar{h}_{dd}(d;0) = d^{q-1}p\begin{bmatrix} -d^{p-q}(p+1)qk \\ +d^{p-q-1}(p-1)(q+1) + q(q+1)(k-1) \end{bmatrix}. \tag{B.7}$$

Let us investigate the function $l(\,\cdot\,)$,

$$l(d) = -d^{p-q}(p+1)qk + d^{p-q-1}(p-1)(q+1) + q(q+1)(k-1). \tag{B.8}$$

Its derivative is

$$l_d(d) = d^{p-q-2}[-d(p-q)(p+1)qk + (p-q-1)(p-1)(q+1)]. \tag{B.9}$$

Let us denote by $\bar{d}$ the positive root of function (B.9)

$$\bar{d} = \frac{(p-q-1)(p-1)(q+1)}{(p-q)(p+1)qk}. \tag{B.10}$$

Suppose first that $\bar{d} \geq 1$. We have that the derivative $l_d(d)$ is positive in the interval $(0, 1)$ and therefore the function $l(d)$ is increasing. Since $l(0) > 0$, the function $l(d)$ is positive. Hence, the derivative $\bar{h}_d(d;0)$ is increasing. Since $\bar{h}_d(1;0) = 0$, it is negative in the interval $(0, 1)$. Therefore $\bar{h}_d(d;\xi) < \bar{h}_d(d;0) < 0$. So, the function $\bar{h}(d;\xi)$ is decreasing. Since $\bar{h}(0;\xi) = p-q-1 > 0$ and $\bar{h}(1;\xi) = -\xi p < 0$, the equation $\bar{h}(d;\xi) = 0$ has only one root in the interval $(0, 1)$.

Suppose now that $\bar{d} < 1$. Therefore $l_d(d) > 0$ for $d < \bar{d}$ and $l_d(d) < 0$ for $d > \bar{d}$. Since $l(0) > 0$, the function $l(d)$ starts from the positive value, increases to a maximum and then decreases. If $l(1) \geq 0$, then the previous case holds. Suppose that $l(1) < 0$. Hence, the function $\overline{h}_{dd}(d; 0)$ is first positive and after that negative. So, the function $\overline{h}_d(d; 0)$ starts from the negative value $\overline{h}_d(0; 0) = -k(p-q)$ increases to a positive maximum and then decreases to zero staying positive. Therefore the function $\overline{h}(d; 0)$ starts from the positive value $p-q-1$ decreases to a negative minimum and then increases to zero. We know that if the function $\overline{h}(d; 0)$ decreases, then $\overline{h}(d; \xi)$ decreases faster, since $\overline{h}_d(d; \xi) < \overline{h}_d(d; 0)$. Also, if $\overline{h}(d; 0)$ increases, then $\overline{h}(d; \xi)$ increases slower or decreases. Thus we can conclude that the equation $\overline{h}(d; \xi) = 0$ has only one root in the interval (0, 1).

Therefore the equation $h(b; \xi) = 0$ has only one root larger than one, since $b = \frac{1}{d}$.

B2. Buyer's boundary

We shall prove that derivative (3.9) has only root in the interval (0, 1). Let $h(\,\cdot\,;\ \cdot\,)$ be the function

$$h(a; \xi) = -a^{p+1}(p-q-1) + a^p k(p-q) \\ -a^{p-q}p(k-1+\xi) - a(q+1) + qk. \tag{B.11}$$

We investigate the undiscounted case $p = q+1$ first. The function (B.11) transforms to

$$h(a; \xi) = a^p k - ap(k+\xi) + (p-1)k. \tag{B.12}$$

Its derivative

$$h_a(a; \xi) = p\left(a^{p-1}k - k - \xi\right) \tag{B.13}$$

is negative in the interval (0, 1) and therefore the function $h(a; \xi)$ is decreasing. Since $h(0; \xi) = (p-1)k > 0$ and $h(1; \xi) = -\xi p < 0$, the equation $h(a; \xi) = 0$ has a unique root.

Assume now that $p > q+1$ and therefore the derivative of function (B.11) transforms to

$$h_a(a; \xi) = -a^p(p+1)(p-q-1) + a^{p-1}pk(p-q) \\ -a^{p-q-1}(p-q)p(k-1+\xi) - (q+1). \tag{B.14}$$

We shall investigate the case $\xi = 0$. We have $h_a(a;\ \xi) < h_a(a;\ 0)$. The second derivative of function (B.11) is

$$h_{aa}(a; 0) = a^{p-q-2}p \\ \times \begin{bmatrix} -a^{q+1}(p+1)(p-q-1) + a^q(p-1)k(p-q) \\ -(p-q-1)(p-q)(k-1) \end{bmatrix}. \tag{B.15}$$

Let the function $l(a)$ be

$$l(a) = -a^{q+1}(p+1)(p-q-1) + a^q(p-1)k(p-q) \\ -(p-q-1)(p-q)(k-1). \tag{B.16}$$

It has a derivative

$$l_a(a) = a^{q-1}[-a(q+1)(p+1)(p-q-1) + q(p-1)k(p-q)] \tag{B.17}$$

which root is

$$\overline{a} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}. \tag{B.18}$$

Suppose that $\overline{a} \geq 1$. Therefore the derivative $l_a(a)$ is positive in the interval (0, 1). Hence, the function $l(a)$ is increasing. We have that $l(0) < 0$. Suppose that $l(1) < 0$. Then $h_a(a;\ 0)$ is decreasing and therefore it is negative, since $h_a(0; 0) = -(q+1) < 0$.[4] Otherwise, if $l(1) > 0$, then the function $h_a(a;\ 0)$ starts from the negative value $h_a(0; 0) = -(q+1)$, has a minimum and then increases to zero. Therefore $h_a(a;\ 0) < 0$ and thus the derivative $h_a(a;\ \xi)$ is negative too. Hence the equation $h(a; \xi) = 0$ has only one root since $h(0; \xi) = qk > 0$ and $h(1; \xi) = -\xi p < 0$.

Suppose now that $\overline{a} < 1$. Then the function $l(a)$ starts from a negative value, increases to a maximum for $a = \overline{a}$ and then decreases. If $l(\overline{a}) < 0$ or $l(1) > 0$, the previous case holds. Suppose that $l(\overline{a}) > 0$ and $l(1) < 0$. Therefore the derivative $h_a(a;\ 0)$ starts from the negative value $h_a(0; 0) = -(q+1)$, decreases to a negative minimum, increases to a positive maximum and after that decreases to zero. In that way $h_a(a;\ 0)$ is first negative and then positive. Therefore the function $h(a;\ 0)$ starts from the positive value $h(0; 0) = qk$, decreases to a negative minimum, and increases to zero after that. The fact that $h_a(a;\ \xi) < h_a(a;\ 0)$ means that when the function $h(a;\ 0)$ decreases, the function $h(a;\ \xi)$ decreases faster. Also, if $h(a;\ 0)$ increases, then $h(a;\ \xi)$ increases slower or decreases. Thus we can see that the function $h(a;\ \xi)$ has only one root.

[4] In fact this case is possible only if $\xi > 0$.

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